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RF-3616

M. Sc. (Part - II) Examination

April / May – 2010

Mathematics : Paper - 5008

(Advanced Special Functions)

Time : Hours] [Total Marks :

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **M. Sc. (Part - 2)**

Name of the Subject :
☛ **Mathematics-5008**

☛ Subject Code No. : **3 6 1 6** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Attempt all questions.

(3) Figure to the right indicates marks.

(4) Notations and conventions are all standard.

1 (a) If $a+b+\frac{1}{2}$ is neither zero nor negative integer and if $|x|<1$ and $|4x(1-x)|<1$ then show that **8**

$$F\begin{bmatrix} a, b\,; & \\ a+b+\frac{1}{2}; & 4x(1-x) \end{bmatrix} = F\begin{bmatrix} 2a,\ 2b; & \\ a+b+\frac{1}{2}; & x \end{bmatrix}$$

(b) Define contiguous function to $F(a,b;c;z)$ Derive the relations : **6**

(i) $(a-b)F = aF(a+) - bF(b+)$

(ii) $[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)z\,F(Cc+)$

OR

1 (a) If $|z| < 1$ and $|1 - z| < 1$, $\text{Re}(c) < 1, \text{Re}(c - a - b) > 0$ then prove that : **8**

$$F(a, b\,; a + b + 1 - c; 1 - z)$$

$$= \frac{\Gamma(a + b + 1 - c)\,\Gamma(1 - c)}{\Gamma(a + 1 - c)\,\Gamma(b + 1 - c)}\ F(a, b\,; c; z)$$

$$+ \frac{\Gamma(a + b + 1 - c)\,\Gamma(c - 1)}{\Gamma(a)\,\Gamma(b)}\ z^{1-c} F(a + 1 - c, b + 1 - c; 2 - c; z)$$

(b) If $k(k)$ is the complete elliptic integral of the second kind, then prove that **6**

$$\int_0^t k\left(\sqrt{x(t - x)}\right) dx \ = \pi \sin^{-1}\left(\frac{t}{2}\right)$$

2 (a) Define contiguous function to ${}_pF_q$ obtain the relation **8**

$$\alpha_1 F = \alpha_1 F(\alpha_1 +) - x \sum_{j=1}^{q} U_j\ F(\beta_j +),\ p < q$$

$$\theta F(\alpha_k -) = (\alpha_k - 1)\, x \sum_{j=1}^{q} w_{j,k}\ F(\beta_j +),\ p \le q$$

(b) If n is a non-negative integer and if a, b, c are independent of n, prove that : **6**

$${}_3F_2\left[\begin{matrix} -n, a, b; \\ c, 1 - c + a + b - n\,; \end{matrix}\ 1\right] = \frac{(c - a)_n\,(c - b)_n}{(c)_n\,(c - a - b)_n}$$

OR

2 (a) If neither $(a-b)$ nor $(a-c)$ nor a is a negative integer then prove that **8**

$$ {}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix} \; x\right] $$

$$ = (1-x)^{-a} \; {}_3F_2\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, & 1+a-b-c; \\ 1+a-b, & 1+a-c; \end{matrix} \; \frac{-4x}{(1-x)^2}\right] $$

(b) If $p = q+1$ and no a_m is zero or a negative integer then show that **6**

$$ \frac{1}{2\pi i} \int_B \frac{\overline{|(c-s)}\,(-z)^s \prod_{m=1}^{p} \sqrt{(a_m+s)}}{\prod_{j=1}^{q} \overline{|(b_j+s)}} $$

where B is Barnes contour, is an analytic function ( )f z in the plane $|\arg(-z)| < \pi$.

3 (a) State and prove the Kummer's second formula. **8**

(b) The error function $erf(x)$ is defined by **6**

$$ erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt, \text{ then show that} $$

$$ erf(x) = \frac{2x}{\sqrt{\pi}} \; {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x^2\right) $$

OR

3 (a) Let $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$, $\gamma_0 \neq 0$. And the polynomials **8**

$f_n(x)$ are defined by $(1-t)^{-c}\, \psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x) t^n$

prove that

$$xf_n'(x) - nf_n(x) = \sum_{k=0}^{n-1} (-1)^{n-k} (c+2k) f_k(x), n \geq 1$$

(b) For the polynomials $P_n(x)$ defined by **6**

$A(t)\, \psi(xH(t)) = \sum = \sum_{n=0}^{\infty} P_n(x) t^n$ with

$\psi(t) = \sum_{n=0}^{\infty} \gamma_n t^n, A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}$

holding $(\gamma_0, a_0, h_0 \text{ are not zero})$ then there exists a sequence of number α_k and β_k such that for $n \geq 1$

$$xP_n'(x) - nP_n(x) \equiv -\sum_{k=0}^{n-1} \alpha_k P_{n-1-k}(x) - x \sum_{k=0}^{n-1} \beta_k P_{n-1-k}'(x)$$

Provide $t\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1}$ and

$t\frac{H'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1}$.

4 (a) Define simple sets of polynomials. Let $\{\phi_n(x)\}$ is a simple set of polynomials and if $P(-x)$ is a polynomial of degree m then show that there exists constants c_k such that 8

$$p(x)=\sum_{k=0}^{m} c_k\,\phi_k(x)$$

In which c_k are functions of k and of any parameters involved in $p(x)$.

(b) State and prove the necessary and sufficient condition for the set $\phi_n(x)$ be orthogonal w.r.t. $w(x)$ over the interval $a<x<b$ 6

OR

4 (a) Let $\phi_n(x)=h_n x^n+\prod_{n-1}$ be a simple set of real polynomials orthogonal with respect to $w(x)>0$ on $a<x<b$. Show that $\sum_{k=0}^{n} g_k^{-1}\,\phi_k(x)\,\phi_k(y)$ 8

$$=\frac{h_n}{g_n h_{n+1}}\left[\frac{\phi_{n+1}(y)\,\phi_n(x)-\phi_{n+1}(x)\,\phi_n(y)}{y-x}\right]$$

(b) With usual notation prove that : **6**

(i) $P_n^{(\alpha,\beta)}(1) = 1$

(ii) $P_n^{(\alpha,\beta)}(x) = \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, -\alpha-n; & \frac{x+1}{x-1} \\ 1+\beta; & \end{matrix}\right]$

5 (a) With usual notation prove that **8**

$$\sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_n \, P_n^{(\alpha,\beta)}(x)\, t^n}{(1+\alpha)_n}$$

$$= (1-t)^{-1-\alpha-\beta} \, {}_2F_1\left[\begin{matrix} \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; & \frac{2t(x-t)}{(1-t)^2} \\ 1+\alpha \qquad ; & \end{matrix}\right]$$

(b) Obtain the differential recurrence relation **6**

$$(x-1)\left[(\alpha+\beta+n) DP_n^{(\alpha,\beta)}(x) + (\alpha+n)\, DP_{n-1}^{(\alpha,\beta)}(x)\right]$$

$$= (\alpha+\beta+n)\left[n\, P_n^{(\alpha,\beta)}(x) + (\alpha+n)\, DP_{n-1}^{(\alpha,\beta)}(x)\right]$$

OR

5 (a) Prove that : **8**

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left\{P_n^{(\alpha,\beta)}(x)\right\}^2 dx$$

$$= \frac{2^{1+\alpha+\beta}\, \Gamma(1+\alpha+n)\, \Gamma(1+\beta+n)}{n!\,(1+\alpha+\beta+2n)\, \Gamma(1+\alpha+\beta+n)}$$

(b) With usual notation prove that **6**

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)} x\, t^n = \rho^{-1}\left(\frac{2}{1+t+\rho}\right)^{\beta}\left(\frac{2}{1-t+\rho}\right)^{\alpha}$$

where $\rho = \left(1 - 2xt + t^2\right)^{1/2}$.